

Representations of the quantum torus and applications to finitely presented groups

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1 Introduction

The first strand of this paper concerns the (algebraic) quantum torus. By this we mean the crossed product FA of a field F with a free abelian group A of finite rank, and much of the time we shall assume that F is the centre of FA .

The representation theory of such crossed products depends heavily on the image in the multiplicative group of F of the 2-cocycle used to define the crossed product. For example in [7] the first author showed that the global dimension is equal to the maximal rank of a subgroup B for which the sub-crossed product FB is commutative. It is natural to concentrate on impervious FA -modules, those non-zero modules that contain no non-zero submodule induced from a module over a sub-crossed product FB for a subgroup B of infinite index in A . (A consequence of this condition when F is the centre of FA is that M is FB -torsion-free for every subgroup B of A for which the sub-crossed product FB is commutative.)

In [7] it is shown that, under the assumption that the centre of FA is exactly F , the Gelfand-Kirillov dimension, $\dim M$, of any impervious FA -module M is at least one half of the rank of A . This is analogous to Bernstein's inequality for Weyl algebras. Following the terminology used for Weyl algebras, we shall say that an FA -module M is *strongly holonomic* if it is impervious and $\dim M$ is exactly one half of the rank of A . We prove the following:

Theorem (Theorem 4.2). *Let M be a strongly holonomic FA -module. Then there is a subgroup of finite index in A having the form*

$$A_1 \oplus \cdots \oplus A_t$$

so that if $i \neq j$ then FA_i commutes with FA_j and the 2-cocycles defining the crossed products FA_i have infinite cyclic image in the multiplicative group of F , and these images of 2-cocycles are non-commensurable if $i \neq j$. Further, if we consider M as FA_i -module then there exist FA_i -submodules which are strongly holonomic.

The second strand of the paper is to consider the structure of certain finitely presented groups. Twenty five years ago it seemed likely that an understanding of finitely presented abelian-by-nilpotent groups would quickly follow the understanding of finitely presented metabelian groups gained in Baumslag [2] and in Bieri and Strebel [4].

It soon became clear, however, that even the existence of finitely presented abelian-by-nilpotent groups which were not also nilpotent-by-abelian-by-finite was a non-trivial question. It was settled, however, by Robinson and Strebel in [16]. They provided examples where the ‘nilpotent top’ was either a Heisenberg group of Hirsch length 3 or a direct product of such a group with an infinite cyclic group. It is not difficult to extend their techniques to provide examples with ‘nilpotent top’ which are Heisenberg of any rank or, more generally, a central product of such groups and cyclic groups. (Here a Heisenberg group is one of the form

$$\langle x_1, \dots, x_n, y_1, \dots, y_n, z : [x_i, x_j] = [y_i, y_j] = [x_i, z] = [y_i, z] = 1, [x_i, y_i] = z \rangle$$

where the indices i, j are allowed to run between 1 and n ; and by ‘nilpotent top’ we mean the quotient by the Fitting subgroup.)

In [11], the first author, Roseblade and Wilson showed that a finitely presented abelian-by-polycyclic group is virtually nilpotent-by-nilpotent. In [7], Brookes showed that a finitely presented abelian-by-nilpotent group is virtually nilpotent-by-nilpotent of class 2. Thus the question of finitely presented abelian-by-polycyclic groups essentially reduces to the case that the ‘polycyclic top’ is nilpotent of class 2.

We argue that the restriction is, in some sense, greater still. In fact, with a natural restriction, we are reduced to examples which have a similar structure to the generalisations of the Robinson-Strebel examples mentioned above. We cannot in this case restrict the Fitting quotient in general because it turns out that the set of possible Fitting quotients is closed under subdirect product and hence includes all finitely generated nilpotent groups of class 2. However we do make progress if we focus on subdirectly irreducible finitely presented groups, those where any two non-trivial normal subgroups have non-trivial intersection.

Theorem (Corollary 5.5). *Let G be a finitely presented group which is an extension of an abelian by a group which is torsion-free nilpotent of class two. Suppose that G is subdirectly irreducible. Then the quotient by the Fitting subgroup of G has a subgroup of finite index which is a central product of groups which are either Heisenberg or cyclic.*

It seems likely that the same will be true if one weakens ‘finitely presented’ by replacing it with the condition that G is the quotient of a small finitely presented group (i.e. one without free subgroups of rank two).

After some definitions and basic results in Section 2, we consider some properties of the geometric invariant, introduced in [8], for modules over crossed products. This invariant is related to one used by Bieri and Strebel in the classification of finitely presented metabelian groups [4], which in turn was related to the logarithmic limit set of Bergman [1]. Such invariants are also of interest in tropical geometry [12]. In section 3 we consider local cones and provide an alternative proof to Theorem B of Wadsley [19] linking them with trailing coefficient modules. In Section 4, we prove the main result about strongly holonomic modules and complete the paper by applying this result to groups in Section 5.

2 Definitions and preliminary results

Throughout this section and the next, D will denote a division ring, A a finitely generated free abelian group and DA a crossed product of D with A . By the *rank* of any abelian group B , we shall always mean the torsion-free

rank or, equivalently, the \mathbb{Q} -dimension of the tensor product $B \otimes \mathbb{Q}$ of B with the rational numbers \mathbb{Q} . We shall denote the rank of B by $\text{rk } B$. The rank of A will always be denoted by n . All modules will be right modules. All functions will be written on the left.

The structure of DA demands that there is a D -vector space basis \bar{A} of DA , consisting of units and in bijective correspondence $a \rightarrow \bar{a}$ with A , and it is convenient to assume throughout that $\bar{1}$ is the multiplicative identity of DA ; thus each element α of DA can be uniquely expressed as a sum of the form

$$\alpha = \sum \bar{a} d_a$$

with $d_a \in D$, $\bar{a} \in \bar{A}$ and only finitely many d_a non-zero. We shall refer to the finite set of elements $a \in A$ such that d_a is non-zero as the *support* of α , written $\text{Supp}(\alpha)$, and for a subset X of A write DX for the set of elements of DA with support in X . The multiplication in DA depends on a 2-cocycle with image in the multiplicative group of D .

If B is a submonoid of A then DB is a subring which has a natural structure as a crossed product of D with B . Because A is torsion-free abelian of finite rank, it is orderable and so it is easy to prove that DA has no non-zero divisors of zero. Further details, as well as proofs of some of the statements made here, can be found in the book by Passman [15].

We denote the homomorphism group $\text{Hom}(A, \mathbb{R})$ by A^* and use similar notation for other abelian groups. As A has finite rank, $\chi(A)$ will also have finite rank; we call this the *rank* of χ . We often extend the definition of χ to DA by defining, for non-zero $\alpha \in DA$, $\chi(\alpha) = \min\{\chi(a)\}$ where a is allowed to run through the support of α . If B is a subgroup of A then there is a natural map from A^* to B^* obtained by restriction. We denote this map by π_B . If C is a subgroup of B then the corresponding map from B^* to C^* is denoted by π_C^B .

We shall denote by M a finitely generated DA -module. Then M defines a subset $\Delta(M)$ of A^* . We refer to [9] for a full definition but the most useful characterisation for the current purposes is the following.

$$\begin{aligned} \chi \notin \Delta(M) \text{ if and only if, for each } m \in M, \text{ there is a relation} \\ m \cdot (1 + \alpha) = 0 \text{ with } \alpha \in DA \text{ and } \chi(a) > 0 \text{ for each } a \text{ in the} \\ \text{support of } \alpha. \end{aligned} \tag{2.1}$$

We refer to [9, Section 3] for a fuller discussion of the elementary properties of $\Delta(M)$.

The *dimension* of M is the largest natural number m so that M contains a non-zero torsion-free DB -submodule for some $B \leq A$ with B of rank m . The properties of this dimension are discussed in [9]; in particular, it is shown that it coincides with the standard Gelfand-Kirillov dimension.

It turns out that it is much easier to describe a large subset of $\Delta(M)$. A point x of $\Delta(M)$ is *regular* if some neighbourhood of x in $\Delta(M)$ is an m -ball for some positive integer m and if m is the largest integer for which this can occur. Then $\Delta^*(M)$ is the Euclidean closure of the set of regular points of $\Delta(M)$. The main result of [9] was that, if M has dimension m , then $\Delta^*(M)$ is a rational polyhedron and that the points lying in $\Delta(M)$ but not in $\Delta^*(M)$ can be enclosed within a rational polyhedron of dimension $m - 1$. (Here a *rational polyhedron* is a finite union of finite intersections of half-spaces with boundaries defined by a linear equation with rational coefficients. It is of *dimension* m if it contains m -balls but no k -balls for $k > m$.) Recently Wadsley [19] has shown that $\Delta(M)$ is itself polyhedral.

The local cone was introduced, in the case of DA commutative, in [3] in an attempt to describe local behaviour of Δ . Let $S \subseteq A^*$ and let $x \in S$. The *local cone* of S at x is

$$LC_x(S) = \{y : \text{for some } \epsilon_0 > 0, x + \epsilon y \in S \text{ for all } \epsilon \in [0, \epsilon_0]\}.$$

Observe that the local cone is a cone, centered at the origin. In all cases here, S will be either $\Delta(M)$ or $\Delta^*(M)$. The dimension of $LC_\chi(\Delta(M))$, for a regular point χ , equals the dimension of $\Delta(M)$ and so that of M .

In this and the next section, we shall be interested in the relation between the concept of local cone and the following concept, which we can also regard as being ‘local’.

Definition 1. *Let M be a finitely generated DA -module furnished with a finite generating set \mathcal{X} . Fix $\chi \in A^*$ and set $A(0) = \{a \in A : \chi(a) \geq 0\}$ and $A(+) = \{a \in A : \chi(a) > 0\}$. Then $A(0)$ and $A(+)$ are subsemigroups of A and we can form the sub-crossed products $DA(0)$ and $DA(+)$. Define the trailing coefficient module $TC_\chi(M)$ to be*

$$TC_\chi(M) = \mathcal{X}.DA(0)/\mathcal{X}.DA(+);$$

it is naturally a module for DB where B is the kernel of χ .

Observe that, using the characterisation of $\Delta(M)$ above, it follows immediately that if $\chi \notin \Delta(M)$ then $TC_\chi(M)$ is zero. The converse follows from Proposition 3.1 of [9]. Because $TC_\chi(M)$ is a DB -module it again has a Δ -set, which is a subset of B^* .

Theorem B of [19] establishes the relationship between the local cone at χ and $\Delta(TC_\chi(M))$. The main aim of the rest of this section and the next is to provide an alternative approach to that result.

First we wish to establish a useful technical condition for inclusion in $\Delta(TC_\chi(M))$. We have extracted part of the proof of this as a technical lemma. It enables us to apply results which are standard for Noetherian rings to non-Noetherian subrings of DA .

Lemma 2.1. *Let U and V_1 be subsemigroups of A and V a submonoid of A with $UV \subseteq U, VV_1 \subseteq V_1$ and $V_1 \subseteq V$. Then*

1. $R = DU + DV$ is a subring of DA and $J = DU + DV_1$ is an ideal of R ;
2. $1 - J$ is a right denominator set in R (in the sense of [14, 2.1.13]);
3. if each $x \in \mathcal{X}$ is $(1 - J)$ -torsion then so also is each $m \in M$.

Proof. The first statement of the lemma is a routine check.

Set $T = 1 - J$. We show that T is a right Öre set in R . Recall that this means that we must show that, if $r \in R$ and $t \in T$ then there exist elements $r' \in R$ and $t' \in T$ so that $rt' = tr'$.

The union of the supports of r and t is finite and so we can find finitely generated subsemigroups \widehat{U} of U , \widehat{V}_1 of V_1 and a finitely generated submonoid \widehat{V} of V so that $r \in D\widehat{U} + D\widehat{V}$ and $t \in D\widehat{U} + D\widehat{V}_1$. Set $\tilde{U} = \widehat{U}\widehat{V}$, $\tilde{V} = \widehat{V}$ and $\tilde{V}_1 = \widehat{V}\widehat{V}_1$. Then \tilde{U}, \tilde{V}_1 are subsemigroups and \tilde{V} is a submonoid; further, $\tilde{U}\tilde{V} \subseteq \tilde{U}$, $\tilde{V}\tilde{V}_1 \subseteq \tilde{V}_1$ and $\tilde{V}_1 \subseteq \tilde{V}$. Also

$$r \in \tilde{R} = D\tilde{U} + D\tilde{V} \text{ and } t \in \tilde{J} = D\tilde{U} + D\tilde{V}_1.$$

As in part (1) of the lemma, \tilde{R} is a ring with ideal \tilde{J} .

We can use a non-commutative version of the Hilbert basis theorem (see, for example, Theorem 10.2.6 of [PassGR]) to show that \tilde{R} is Noetherian. Also, \tilde{J} is generated as ideal of \tilde{R} by elements of \tilde{A} and if $a \in A$ then $\tilde{a}\tilde{R} = \tilde{R}\tilde{a}$. Thus we can apply Proposition 2.6 of [14] and then Proposition 4.2.9 of [14] to show that $1 - \tilde{J}$ is a right Öre set. Thus we can find $r' \in \tilde{R}$ and $t' \in 1 - \tilde{J}$ so that $rt' = tr'$. As $r' \in R$ and $t' \in 1 - J$, this shows, therefore, that T is a right Öre set. Because DA has no divisors of zero, neither does R and so T is a right denominator set.

To prove the last part of the lemma, let $m \in M$ and suppose that $m = \sum x d \tilde{a}$ with $x \in \mathcal{X}$, $d \in D$ and $a \in A$. Suppose that, for $x \in \mathcal{X}$, we have $x t_x = 0$ with $t_x \in T$. Then $(x d \tilde{a}) \cdot (t_x)^{d \tilde{a}} = 0$ and $(t_x)^{d \tilde{a}}$ is still an element of J . It is a standard check, using the right Öre condition, that a sum of T -torsion elements is still T -torsion and so m is also T -torsion. \square

Lemma 2.2. *Let $\chi \in A^*$ and let B denote the kernel of χ . Let $\psi \in B^*$. Then $\psi \notin \Delta(TC_\chi(M))$ if and only if*

$$\begin{aligned} &\text{for each } m \in M, \text{ there exist } \alpha \in DA \text{ and } \beta \in DB \text{ with} \\ &m = m\alpha + m\beta \text{ and } \chi(\alpha) > 0 \text{ and } \psi(\beta) > 0. \end{aligned} \quad (2.2)$$

Proof. Let R denote $DA(+) + DB(0)$ and let J denote $DA(+) + DB(+)$ of R . Applying Lemma 2.1 with $U = A(+)$, $V = B(0)$ and $V_1 = B(+)$, we see that R is a subring of DA and J is an ideal of R . Further $1 - J$ is a right denominator set in R .

Applying the definition of the Delta sets, $\psi \notin \Delta(TC_\chi(M))$ if and only if for each $u \in TC_\chi(M)$ there exists $\beta \in DB(+)$ with $u \cdot (1 + \beta) = 0$. Applying the definition of the trailing coefficient module, this implies

$$\begin{aligned} &\text{for each } x \in \mathcal{X}, \text{ there exist } \beta \in DB(+), x_i \in \mathcal{X} \text{ and } \rho_i \in \\ &DA(+) \text{ so that } x \cdot (1 + \beta) = \sum_i x_i \rho_i. \end{aligned} \quad (2.3)$$

Reversing this last argument, we see that $\psi \notin \Delta(TC_\chi(M))$ if and only if (2.3) holds.

Let N denote the R -submodule of M generated by \mathcal{X} . If (2.3) holds then for each $x \in \mathcal{X}$, we have $x \in NJ$ from which it follows easily that $N = NJ$ and reversing the argument shows that $N = NJ$ is equivalent to (2.3). Thus $\psi \notin \Delta(TC_\chi(M))$ if and only if $N = NJ$.

Observe that (2.2) is equivalent to the condition that M is $(1 - J)$ -torsion and from (3) of Lemma 2.1 this is equivalent to the condition that N is $(1 - J)$ -torsion. Thus the lemma is reduced to showing that N is $(1 - J)$ -torsion if and only if $N = NJ$.

If N is $(1 - J)$ -torsion, then clearly $N = NJ$. For the converse observe that, since $T = 1 - J$ is a right denominator set in R we can form the ring of quotients R_T and the module of quotients N_T . Then N_T is a finitely generated R_T module satisfying $N_T = N_T J_T$. But J_T lies in the Jacobson radical of R_T and so, by Nakayama's lemma (see, for example 0.3.10 of [MR]), $N_T = 0$. That is N is T -torsion, as required. \square

Observe that the proof of the lemma shows that it is sufficient, in (2.2), to assume that the condition holds for all m belonging to some generating set of M .

Lemma 2.3. *Suppose that $L \rightarrow M \rightarrow N$ is a short exact sequence of DA -modules. Then*

$$\Delta(TC_\chi(M)) = \Delta(TC_\chi(L)) \cup \Delta(TC_\chi(N)).$$

Proof. This is an immediate application of Lemma 2.2. \square

Much of the content of the next two sections will be to relate the local cone to the Delta set of the trailing coefficient module. We begin with a relatively simple observation.

Lemma 2.4. *Let $\chi \in A^*$ and let B denote the kernel of χ . Then*

$$LC_{\Delta(M)}(\chi) \subseteq \pi_B^{-1}(\Delta(TC_\chi(M))).$$

Proof. Suppose that $\psi \in A^*$ with $\psi|_B = \pi_B(\psi) \notin \Delta(TC_\chi(M))$. We must show that $\psi \notin LC_{\Delta(M)}(\chi)$.

Since $\psi|_B \notin \Delta(TC_\chi(M))$ then for each $x \in \mathcal{X}$ there exists $\beta_x \in DB$ with $(x + \mathcal{X}.DA(+)) \cdot \beta_x = \{0\}$ and $\beta_x = 1 + \gamma_x$ with $\psi(\gamma_x) > 0$. This implies that

$$x \cdot (1 + \gamma_x) = \sum_{y \in \mathcal{X}} y \alpha_{x,y}$$

with $\alpha_{x,y} \in DA$ and $\chi(\alpha_{x,y}) > 0$.

Choose ϵ_0 by

$$0 < \epsilon_0 < \min \frac{\chi(a)}{|\psi(a)|}$$

where a is allowed to range through the support of all the elements $\alpha_{x,y}$ with $x, y \in \mathcal{X}$. If $0 < \epsilon \leq \epsilon_0$ then, for a in the support of some $\alpha_{x,y}$

$$\chi(a) + \epsilon\psi(a) > 0.$$

Also, for b in the support of some γ_x ,

$$(\chi + \epsilon\psi)(b) = 0 + \epsilon\psi(b) > 0.$$

Thus, for each $x \in X$ we have an expression of the form

$$x = \sum_{y \in X} y \cdot \delta_{x,y}$$

with $\delta_{x,x} = \alpha_{x,x} - \beta_1$ and, if $x \neq y$, $\delta_{x,y} = \alpha_{x,y}$. Thus $(\chi + \epsilon\psi)(\delta_{x,y}) > 0$. It follows that $\chi + \epsilon\psi \notin \Delta(M)$ for $0 < \epsilon < \epsilon_0$. Thus $\psi \notin LC_{\Delta(M)}(\chi)$, as required.

□

Lemma 2.5. *Let $\chi \in A^*$ and let B denote the kernel of χ . Then*

$$\text{rk}(\chi) + \dim_{DB}(TC_\chi(M)) \leq \dim_{DA}(M).$$

Proof. Pick an isolated subgroup B_1 of B of rank equal to the dimension of $TC_\chi(M)$ so that, for some $\bar{y} \in TC_\chi(M)$, we have $\bar{y}.DB_1 \cong DB_1$. Now choose an isolated subgroup C of A so that $B + C = A$ and so that $C \cap B = B_1$. Choose $y \in M$ so that y has the image \bar{y} in $TC_\chi(M)$. We claim that $y.DC \cong DC$. If this claim is true, then

$$\begin{aligned} \dim(M) \geq \text{rk}(C) &= \text{rk}(A/B) + \text{rk}(B_1) \\ &= \text{rk}(A/B) + \dim(TC_\chi(M)) \\ &= \text{rk}(\chi) + \dim(TC_\chi(M)). \end{aligned}$$

(Recall that $\text{rk}(\chi) = \text{rk}(\text{im}(\chi)) = \text{rk}(A/\ker(\chi))$.)

If $y.DC$ is not isomorphic to DC then there is some element $\alpha \in DC$ such that $y.\alpha = 0$. By multiplying α , if necessary, by a suitable element

of C , we can assume that $\alpha = \alpha_0 + \alpha_1$ with $\chi(a) = 0$ for every a in the support of α_0 and $\chi(\alpha_1) > 0$ and $\alpha_0 \neq 0$. Thus $\alpha_0 \in DB \cap DC = DB_1$. Passing to $TC_\chi(M)$, we have $\bar{y}.\alpha_0 = 0$. This contradicts the assumption that $\bar{y}.DB_1 \cong DB_1$ and so completes the proof of the claim. \square

Proposition 2.6. *If $\chi \in \Delta^*(M)$ then*

$$\text{rk}(\chi) + \dim_{DB}(TC_\chi(M)) = \dim_{DA}(M).$$

Proof. Suppose that the dimension of M is m . If $\chi \in \Delta^*(M)$ then χ lies in at least one polyhedron of dimension m within $\Delta(M)$ and so $LC_{\Delta(M)}(\chi)$ has dimension m . By Lemma 2.4, $\pi_B^{-1}(\Delta(TC_\chi(M)))$ has dimension at least m and so $TC_\chi(M)$ has dimension at least $m - r$ where r is the dimension of the kernel of π_B . But r is just the rank of χ . Thus

$$\dim(TC_\chi(M)) \geq m - r = \dim M - \text{rk}(\chi).$$

Combining this with Lemma 2.5 gives the result. \square

Lemma 2.7. *Let $\chi \in \Delta^*(M)$ and let B denote the kernel of χ . Then*

$$LC_{\Delta^*(M)}(\chi) \subseteq \pi_B^{-1}(\Delta^*(TC_\chi(M))).$$

Proof. Suppose that $\Delta(M)$ has dimension m . Then $\Delta^*(M)$ is a finite union of convex polyhedra of dimension m . Thus $LC_{\Delta^*(M)}(\chi)$ has dimension m and so $\pi_B(LC_{\Delta^*(M)}(\chi))$ has dimension at least $m - r$ where r is the dimension of the kernel of π_B or, equivalently, the rank of χ . As $\pi_B(LC_{\Delta^*(M)}(\chi)) \subseteq \pi_B(LC_{\Delta(M)}(\chi))$, then, from Lemma 2.4, $\pi_B(LC_{\Delta^*(M)}(\chi))$ is a subset of $\Delta(TC_\chi(M))$ having dimension at least $m - r$. But Proposition 2.6 tells us that the dimension of $\Delta(TC_\chi(M))$ is exactly $m - r$. Thus $\pi_B(LC_{\Delta^*(M)}(\chi))$ is actually a subset of $\Delta^*(TC_\chi(M))$, as required. \square

3 Trailing coefficient modules and local cones

We retain the notation of the previous section. In particular, D is a division ring, A is an abelian group, DA is a crossed product of D by A , $\chi \in A^*$, B is the kernel of χ and M is a DA -module. The aim in this section is to prove

equality in Lemma 2.7. We begin with a simple case in Lemma 3.1 and then proceed to a less restricted case, the ‘co-dimension one’ case, in Lemma 3.3. Then, in Proposition 3.4, we use the fact that an m -dimensional Delta-set can be reconstructed from its projections onto $m + 1$ -dimensional subspaces to reduce the general case to the ‘co-dimension one’ case.

Lemma 3.1. *Suppose that M is a cyclic 1-relator module. Let $\chi \in \Delta^*(M)$ and let B denote the kernel of χ . Then*

$$LC_{\Delta^*(M)}(\chi) = \pi_B^{-1}(\Delta^*(TC_\chi(M))).$$

Proof. If M is a cyclic 1-relator module with relator r , then $\Delta(M)$ is described in Proposition 2.3 of [8]. For each $\chi \in A^*$, write $r = r_\chi + s_\chi$ where if a, b are in the support of r_χ and if c is in the support of s_χ then $\chi(a) = \chi(b) < \chi(c)$. By multiplying r by the inverse of some element of the support of r_χ , we can and will assume that $\chi(a) = 0$ for all a in the support of χ .

Then $\chi \in \Delta(M)$ if and only if the support of r_χ contains more than one element. In this latter case, it is easily verified that $TC_\chi(M)$ (using the same generator as was used for M) is a 1-relator module with relator $r_\chi \in DB$. Thus $\Delta(TC_\chi(M))$ is calculated in an analogous way to that used for $\Delta(M)$; that is, $\psi \in \Delta(TC_\chi(M))$ if and only if $(r_\chi)_\psi$ has support with more than one element. It is an easy consequence of the definition, or the description in [8], that $\Delta(M) = \Delta^*(M)$ for one-relator modules M ; a similar comment then holds for $TC_\chi(M)$.

Suppose that $\psi \in \Delta(TC_\chi(M))$ and choose $\phi \in A^*$ so that $\pi_B(\phi) = \phi|_B = \psi$. Because $\chi(s_\chi) > 0$, we can choose ϵ_0 so that $\chi(c) + \epsilon\phi(c) > 0$ for c in the support of s_χ and $0 < \epsilon \leq \epsilon_0$. Consider $\chi + \epsilon\phi$ for $0 < \epsilon \leq \epsilon_0$. We have that

$$\begin{aligned} r_{\chi+\epsilon\phi} &= (r_\chi)_{\chi+\epsilon\phi} \quad \text{as } (\chi + \epsilon\phi)(s_\chi) > 0 \\ &= (r_\chi)_{\epsilon\phi} \\ &= (r_\chi)_\psi \quad \text{as } r_\chi \in DB. \end{aligned}$$

By assumption, $(r_\chi)_\psi$ has support with more than one element and hence so also does $r_{\chi+\epsilon\phi}$. Thus $\chi + \epsilon\phi \in \Delta(M)$ for $0 < \epsilon \leq \epsilon_0$ and so $\phi \in LC_{\Delta(M)}(\chi)$.

We have thus shown that $\pi_B^{-1}(\Delta^*(TC_\chi(M))) \subseteq LC_{\Delta^*(M)}(\chi)$. The reverse inclusion is provided by Lemma 2.7 and so the proof is complete. \square

Lemma 3.2. *Suppose that A_1 is an isolated subgroup of A and that N is a DA_1 -module. Let $\chi \in A^*$ and let B denote the kernel of χ . Let $\chi_1 = \pi_{A_1}(\chi)$. Then*

$$\Delta(TC_\chi(N \otimes_{DA_1} DA)) = (\pi_{B \cap A_1}^B)^{-1} \Delta(TC_{\chi_1}(N))$$

and so

$$\Delta^*(TC_\chi(N \otimes_{DA_1} DA)) = (\pi_{B \cap A_1}^B)^{-1} \Delta^*(TC_{\chi_1}(N)).$$

Proof. Let $\psi \in B^*$ and suppose that $\psi \notin \Delta(TC_\chi(N \otimes_{DA_1} DA))$. By Lemma 2.2, for each $n \in N$, there exists $\alpha_n \in DA$ and $\beta_n \in DB$ with $n \otimes 1 = (n \otimes 1)(\alpha_n + \beta_n)$ and $\chi(\alpha_n) > 0$, $\psi(\beta_n) > 0$. Fix a transversal \mathcal{T} , containing 1, for A_1 in A ; we can do this so that it contains a transversal for $B \cap A_1$ in B . Each element of the support of α_n can be written uniquely as a product of an element in A_1 and an element in \mathcal{T} . Thus α_n can be written as

$$\alpha_n = \sum_{t \in \mathcal{T}} \alpha_n(t)t \quad \text{with } \alpha_n(t) \in DA_1.$$

Clearly $\chi_1(\alpha_n(1)) > 0$. Similar comments apply for β_n ; in particular $\beta_n(1) \in D(B \cap A_1)$ and $\psi(\beta_n(1)) > 0$. Because $N \otimes_{DA_1} DA$ is an induced module, it follows that $n = n \cdot (\alpha_n(1) + \beta_n(1))$. Thus the restriction of ψ to A_1 , that is $\pi_{B \cap A_1}^B(\psi)$, does not lie in $\Delta(TC_{\chi_1}(N))$.

Suppose, conversely, that $\pi_{B \cap A_1}^B(\psi)$ does not lie in $\Delta(TC_{\chi_1}(N))$. Then, by Lemma 2.2, for each $n \in N$, there exist $\alpha_n \in DA_1$ and $\beta_n \in D(B \cap A_1)$ such that $n = n \cdot (\alpha_n + \beta_n)$ and $\chi_1(a) > 0$ for each a in the support of α_n and $\psi_1(b) > 0$ for each b in the support of β_n . Clearly, $\alpha_n \in DA$ and $\beta_n \in DB$ with $\chi(\alpha_n) > 0$ and $\psi(\beta_n) > 0$. We thus have condition (2.2) holding for those $m \in N \otimes_{DA_1} DA$ of the form $n \otimes 1$. But the latter elements suffice to generate $N \otimes_{DA_1} DA$ as DA -module and so, using the comment at the end of the proof of Lemma 2.2, it follows that $\psi \notin \Delta(TC_\chi(N \otimes_{DA_1} DA))$.

The final equality of the lemma is an immediate deduction. \square

Lemma 3.3. *Suppose that A has rank n and that M is a DA -module of dimension $n - 1$. Let $\chi \in \Delta^*(M)$ and let B denote the kernel of χ . Then*

$$LC_\chi(\Delta^*(M)) = \pi_B^{-1}(\Delta^*(TC_\chi(M))).$$

Proof. Observe firstly that M has a finite series with quotients $\{M_1, \dots, M_s\}$ which are cyclic and critical. Further, as M has dimension $n - 1$, all M_i have dimension at most $n - 1$ and at least one has dimension exactly $n - 1$. By Lemma 2.3,

$$\Delta(TC_\chi(M)) = \cup_{i=1}^s \Delta(TC_\chi(M_i)).$$

By Lemma 2.6, $TC_\chi(M_i)$ has dimension $\dim M_i - \text{rk } \chi$ and hence $\Delta^*(TC_\chi(M_i))$ also has this dimension. Thus $\Delta^*(TC_\chi(M))$ is the union of those $\Delta^*(TC_\chi(M_i))$ for which M_i has dimension $n - 1$. Similarly, $LC_\chi(\Delta^*(M))$ is the union of those $LC_\chi(\Delta^*(M_i))$ for which M_i has dimension $n - 1$. Thus we need prove the result only in case M is cyclic and critical. We shall use an inductive argument on the rank of A .

We deal firstly with the case that there is a subgroup A_1 of A with A/A_1 infinite cyclic so that M is not torsion-free as DA_1 -module. Then the set of DA_1 -torsion elements of M forms a non-zero DA -submodule M_1 of M . Let N be a critical DA_1 -submodule of M_1 . By Lemma 2.4 of [9], $N.DA$ has dimension at most $\dim N + 1$ with equality if and only if $N.DA$ is induced from N . Since A_1 has rank $n - 1$ and N is a torsion DA_1 -module, N has dimension at most $n - 2$. As M is critical, every non-zero submodule, in particular $N.DA$, also has dimension $n - 1$. It follows that $N.DA$ is induced from N , that is $n.DA \equiv N \otimes_{DA_1} DA$, and that N has dimension exactly $n - 2$.

By Corollary 4.5 of [9], $\Delta^*(M) = \Delta^*(N.DA)$ and so, by Lemma 3.4 of [9],

$$\Delta^*(M) = \pi_{A_1}^{-1}(\Delta^*(N)). \quad (3.1)$$

Set $\chi_1 = \pi_{A_1}(\chi) = \chi|_{A_1}$. The inductive argument enables us to assume that

$$LC_{\chi_1}(\Delta^*(N)) = (\pi_{B \cap A_1}^{A_1})^{-1}(\Delta^*(TC_{\chi_1}(N))) \quad (3.2)$$

Observe also that the quotient $M/N.DA$ has smaller dimension than M , because M is critical. By Lemma 2.3 we have

$$\Delta(TC_\chi(M)) = \Delta(TC_\chi(N.DA)) \cup \Delta(TC_\chi(M/N.DA))$$

and the dimensions show that

$$\Delta^*(TC_\chi(M)) = \Delta^*(TC_\chi(N.DA)).$$

We therefore have

$$\begin{aligned}
\pi_B^{-1}(\Delta^*(TC_\chi(M))) &= \pi_B^{-1}(\Delta^*(TC_\chi(N \otimes_{DA_1} DA))) \\
&= \pi_B^{-1}((\pi_{B \cap A_1}^B)^{-1} \Delta^*(TC_\chi(N)) \text{ by Lemma 3.2}) \\
&= \pi_{B \cap A_1}^{-1}(\Delta^*(TC_\chi(N))) \\
&= \pi_{A_1}^{-1}((\pi_{B \cap A_1}^{A_1})^{-1}(\Delta^*(TC_\chi(N)))) \\
&= \pi_{A_1}^{-1}(LC_{\chi_1}(\Delta^*(N)) \text{ by (3.2)}) \\
&= LC_\chi \pi_{A_1}^{-1}(\Delta^*(N)) \\
&= LC_\chi(\Delta^*(M)) \text{ by (3.1).}
\end{aligned}$$

This completes the proof in case M is not torsion-free as DA_1 -module.

Thus we can assume that M is torsion-free as DA_1 -module for each subgroup A_1 with A/A_1 infinite cyclic. Because we are assuming that M is critical, it follows that every proper quotient of M has dimension at most $n-2$ and so must be torsion as DA_1 -module for each subgroup A_1 with A/A_1 infinite cyclic. Thus we have the necessary conditions for Theorem 2.4 of [8] and we can easily deduce from the proof of this theorem that

$$\Delta^*(M) = (\Delta(V_1) \cap \Delta(V_2))^* \quad (3.3)$$

where V_1 and V_2 are 1-relator DA -modules each of which has a quotient isomorphic to M . Using the fact that each module has a quotient isomorphic to M , together with Lemma 2.3, we deduce that

$$\Delta(TC_\chi(M)) \subseteq \Delta(TC_\chi(V_1)) \cap \Delta(TC_\chi(V_2)). \quad (3.4)$$

Observe that (3.3) together with the fact that each $\Delta(V_i)$ has dimension $n-1$, shows that $\chi \in \Delta^*(V_i)$. By Proposition 2.6, each of the three Δ -sets in (3.4) has the same dimension, equal to $(n-1) - \text{rk}(\chi)$. Thus we can replace (3.4) by

$$\Delta^*(TC_\chi(M)) \subseteq (\Delta(TC_\chi(V_1)) \cap \Delta(TC_\chi(V_2)))^*. \quad (3.5)$$

Thus we have

$$\begin{aligned}
LC_\chi(\Delta^*(M)) &= LC_\chi((\Delta(V_1) \cap \Delta(V_2))^*) \\
&= (LC_\chi(\Delta(V_1)) \cap LC_\chi(\Delta(V_2)))^* \text{ using the definition} \\
&\quad \text{of local cones} \\
&= (\pi_B^{-1}(\Delta(TC_\chi(V_1))) \cap \pi_B^{-1}(\Delta(TC_\chi(V_2))))^* \text{ by Lemma 3.1} \\
&= \pi_B^{-1}(\Delta(TC_\chi(V_1)) \cap \Delta(TC_\chi(V_2)))^* \\
&\supseteq \pi_B^{-1}(\Delta^*(TC_\chi(M))) \text{ by (3.5).}
\end{aligned}$$

The reverse inequality has been proved in Lemma 2.7 and so the proof of the lemma is complete. \square

Proposition 3.4. *Suppose that M is a DA -module. Let $\chi \in \Delta^*(M)$ and let B denote the kernel of χ . Then*

$$LC_\chi(\Delta^*(M)) = \pi_B^{-1}(\Delta^*(TC_\chi(M))).$$

Proof. By Theorem 4.4 of [9], we know that $\Delta^*(TC_\chi(M))$ is a rational polyhedron and hence so also is $\pi_B^{-1}(\Delta^*(TC_\chi(M)))$. By Proposition 2.6, we know that the latter has dimension m , say, equal to that of M . Thus $\pi_B^{-1}(\Delta^*(TC_\chi(M)))$ is a finite union of m -dimensional convex polyhedra. The same holds true for $\Delta^*(M)$ and Theorem 4.4 shows that the remainder of $\Delta(M)$ is also contained in a finite union of m -dimensional convex polyhedra. Let $S = \pi_B^{-1}(\Delta^*(TC_\chi(M))) \cup \Delta(M)$.

We wish to apply Theorem 4.3 of [9]. The requirement on S stated in the theorem is satisfied because S itself has been chosen to lie in a polyhedron of dimension m . There therefore exists a finite set \mathcal{X} of m -dimensional subspaces of A^* and a finite set of projections π_i with image A_i^* , with A_i a subgroup of A of rank $m+1$, so that, if we denote by M_i the set M considered as DA_i -module, then

1. $S \subseteq \bigcup_{X \in \mathcal{X}} X$;
2. for each i , $\ker(\pi_i)$ meets each element of \mathcal{X} trivially (in the language of [9], this follows from the fact that π_i is regular with respect to \mathcal{X});

3. $\Delta^*(M) = \cap_i \pi_{A_i}^{-1}(\pi_{A_i}(\Delta^*(M)))$ (this follows from (3) of Theorem 4.3 of [9] in the same way that (4) of Theorem 4.3 of [9] follows from (3)).

Using (1) and (2) above, together with Proposition 3.8 of [9], we see that M_i is finitely generated for each i . By Proposition 3.7 of [9], $\pi_{A_i}(\Delta(M)) = \Delta(M_i)$. Using (2) above, π_{A_i} maps each element of \mathcal{X} faithfully and so $\dim \Delta(M) = \dim \Delta(M_i)$. Thus

$$\pi_{A_i}(\Delta^*(M)) = \Delta^*(M_i). \quad (3.6)$$

Therefore we can replace (3) above by

$$\Delta^*(M) = \cap_i \pi_{A_i}^{-1}(\Delta^*(M_i)). \quad (3.7)$$

Observe that (3.7) implies easily that

$$LC_\chi(\Delta^*(M)) = \cap_i \pi_{A_i}^{-1}(LC_{\chi_i}(\Delta^*(M_i))) \quad (3.8)$$

where χ_i denotes the restriction of χ to A_i . Note that the dimension of M_i is one less than the rank of A_i . Thus we can apply Lemma 3.3 to show that

$$LC_{\chi_i}(\Delta^*(M_i)) = (\pi_{B \cap A_i}^{A_i})^{-1}(\Delta^*(TC_{\chi_i}(M_i))). \quad (3.9)$$

and so, combining (3.8) with (3.9), we have

$$LC_\chi(\Delta^*(M)) = \cap_i (\pi_{B \cap A_i})^{-1}(\Delta^*(TC_{\chi_i}(M_i))). \quad (3.10)$$

A straightforward application of Lemma 2.2 shows that

$$\pi_{B \cap A_i}^B(\Delta(TC_\chi(M))) \subseteq \Delta(TC_{\chi_i}(M_i)). \quad (3.11)$$

We wish to establish the version of (3.11) where the Δ -sets are replaced by their corresponding Δ^* -versions. This is immediate once we know that the dimensions on each side of the equation coincide. We have chosen the subgroups A_i so that the subspaces $\ker(\pi_i)$ meet the elements of \mathcal{X} trivially. In particular, this implies that the $\ker(\pi_i)$ intersect the supporting spaces (that is the spaces spanned by the convex polyhedra making up the Delta set) of $\pi_B^{-1}(\Delta^*(TC_\chi(M)))$ trivially. It follows easily that the kernel of $\pi_{B \cap A_i}^B$ meets each supporting space of $\Delta^*(TC_\chi(M))$ trivially and that

$\ker(\pi_{A_i})$ meets $\ker(\pi_B)$ trivially. Thus $\pi_{B \cap A_i}^B(\Delta(TC_\chi(M)))$ has dimension equal to that of $\Delta(TC_\chi(M))$.

It follows from (3.6) that $\chi_i \in \Delta^*(M_i)$. By Proposition 2.6, the dimension of $\Delta(TC_\chi(M))$ is $m - \text{rk } \chi$ and the dimension of $\Delta(TC_{\chi_i}(M_i))$ is $\dim M_i - \text{rk } \chi_i$. We have already seen that $\dim M = \dim M_i$. Since $\ker(\pi_i)$ meets $\ker(\pi_B)$ trivially, A_i supplements B in A . Recalling that B is the kernel of χ , it follows that $\text{rk } \chi_i = \text{rk } \chi|_{A_i} = \text{rk } \chi$. Thus the dimensions of the two sides of the inequality in (3.11) are equal and so we have

$$\pi_{B \cap A_i}^B(\Delta^*(TC_\chi(M))) = (\pi_{B \cap A_i}^B(\Delta^*(TC_\chi(M))))^* \subseteq \Delta^*(TC_{\chi_i}(M_i)). \quad (3.12)$$

Thus,

$$\begin{aligned} \pi_B^{-1}(\Delta^*(TC_\chi(M))) &\subseteq \cap_i \pi_B^{-1}((\pi_{B \cap A_i}^B)^{-1}(\Delta^*(TC_{\chi_i}(M_i)))) && \text{by (3.12).} \\ &= \cap_i \pi_{B \cap A_i}^{-1}(\Delta^*(TC_{\chi_i}(M_i))) \\ &= LC_\chi(\Delta^*(M)) && \text{by (3.10).} \end{aligned}$$

The reverse inclusion is Lemma 2.7 and so the proof of the proposition is complete. \square

Let C be a subgroup of A and let $V = \ker(\pi_C)$. A character $\chi \in A^*$ is *generic* for V if $\chi \in V$ and $TC_\chi(M)$ is locally of finite dimension as a module for DC . More discussion of genericness can be found in Section 3 of [BG3]

Corollary 3.5. χ is generic for V if and only if $LC_\chi(\Delta^*(M)) \subseteq V$.

Proof. By Proposition 3.4,

$$LC_\chi(\Delta^*(M)) \subseteq V \text{ if and only if } \pi_B^{-1}(\Delta^*(TC_\chi(M))) \subseteq V.$$

But, $\pi_B^{-1}(\Delta^*(TC_\chi(M))) \subseteq V$ if and only if $\ker(\pi_B) \subseteq V$ and $\Delta^*(TC_\chi(M)) \subseteq \pi_B(V)$. But $\ker(\pi_B) \subseteq V$ if and only if $\chi \in V$ and, by Lemma 2.4 of [10], $\Delta^*(TC_\chi(M)) \subseteq \pi_B(V)$ if and only if $\Delta^*(TC_\chi(M))$ is locally of finite dimension as a module for the dual of $\pi_B(V)$. The latter condition is exactly that required for χ to be generic for V and so the proof is complete. \square

Observe that this implies immediately that any point of $\Delta^*(M)$ which is non-generic for some carrier space must lie in at least two distinct carrier

spaces. Proposition 3.6 of [10] guarantees that, under certain technical conditions, a carrier space contains non-generic points. We can use that with Corollary 3.5 to show that, with suitable easily satisfied assumptions, any two carrier spaces of a Delta-set must have non-zero intersection.

4 Modules over central crossed products

4.1 Preliminaries

In this section, F will denote a field, A will denote a free abelian group of rank n and FA will be a crossed product of A with F in which F is central. Thus FA is an A -graded algebra in which each component is isomorphic to F .

Recall from Theorem 4.4 of [BG2] that if M is a finitely generated FA -module then $\Delta^*(M)$ is a finite union of m -dimensional convex polyhedra. The subspace spanned by one of these polyhedra is called a *carrier space* of $\Delta^*(M)$. It is also shown in [BG2] that this subspace is rationally defined and so is the kernel of a projection map $\pi_B : A^* \rightarrow B^*$ where B is a subgroup, which we can assume isolated, of A . We call such a subgroup B a *carrier space subgroup* of M .

Before stating the main result of this section, it will be convenient to collect a couple of results of the first author [7]. They will be key tools in the following.

Proposition 4.1 (Brookes). *Let M be a non-zero finitely generated module over FA . Then*

1. $\dim M = n - \text{rk } B$ for some subgroup B of A with FB commutative.
2. A carrier space subgroup of M has a subgroup B of finite index with FB commutative.
3. If F is the centre of FA and M is impervious then M is FB -torsion-free for every subgroup B of A with FB commutative.

Proof. (1) is Theorem 3 of [7] and (3) is Theorem 4. Part (2) is an immediate consequence of the proof of Theorem 3. Theorem 3 is proved by taking a

character in $\Delta(M)$ of maximal rank m and showing that some subgroup of finite index in the kernel of that character has a subgroup B of finite index with DB commutative. If we start with a carrier space subgroup then this corresponds to a carrier space which must contain a character χ of maximal rank. Then it is easy to show that the kernel of χ is exactly the carrier space subgroup and the claim follows. \square

In the following, $\zeta(A)$ is the largest subgroup of A so that $F\zeta(A)$ is central. Similarly for any subgroup B of A we define $\zeta(B)$ to be the largest subgroup of B for which $F\zeta(B)$ is central in FB .

Definition 2. *Suppose that F is the centre of FA and let M be a finitely generated FA -module. We shall say that M is strongly holonomic if*

1. $\text{rk } A = 2 \dim M$;
2. *if B is any subgroup of A so that FB is commutative, then M is FB -torsion-free.*

The modules that arise from the study of modules over nilpotent groups will have $\zeta(A)$ trivial and will be impervious. But then condition 2 is guaranteed by (3) of Proposition 4.1. Further, (1) and (3) of Proposition 4.1 guarantee that $2 \dim M \geq \text{rk } A$ and we will be able to guarantee, from the finite presentation of the groups from which they arise, that $2 \dim M \leq \text{rk } A$. Thus we also have the first requirement for holonomic.

This section is largely devoted to a proof of the following result:

Theorem 4.2. *Let M be a strongly holonomic FA -module. Then there is a subgroup of finite index in A having the form*

$$A_1 \oplus \cdots \oplus A_t$$

so that

1. *if $i \neq j$ then FA_i commutes with FA_j ;*
2. *the centre of FA_i is F ;*
3. *the image of the 2-cocycle used to define the crossed product each FA_i had infinite cyclic image in the multiplicative group of F .*

4. any two such images of 2 cocylces are not commensurable.

Further, if we consider M as FA_i -module then there exist FA_i -submodules which are strongly holonomic.

We shall henceforth suppose that FA has a strongly holonomic module M . We begin with some of the immediate consequences of this. We shall denote by m the dimension of M .

Lemma 4.3. *Let B be a subgroup of A . Suppose that B contains a subgroup C of rank m with FC commutative and suppose also that*

$$\text{rk } B + \text{rk } \zeta(B) \geq 2m.$$

If we denote by F_1 the field of fractions of $F\zeta(B)$ then equality holds above and there is a crossed product $F_1(B/\zeta(B))$, a localisation of FB , which has a strongly holonomic module.

Proof. Let W denote a non-zero finitely generated FB -submodule of M . Clearly, $\dim_B(W) \leq \dim_A M = m$. But $C \leq B$ and, by supposition, M is FC -torsion-free. Thus $\dim_B W \geq m$ and so $\dim_B W = m$.

Because $F\zeta(B)$ is commutative, M and so W , is torsion-free as $F\zeta(B)$ -module. Set $W_1 = W \otimes_{F\zeta(B)} F_1$. This is a module for $FB \otimes_{F\zeta(B)} F_1$ and it is straightforward to verify that $FB \otimes_{F\zeta(B)} F_1$ is isomorphic to a cross product of F_1 by $B/\zeta(B)$ which we shall denote by $F_1(B/\zeta(B))$. We claim that W_1 is strongly holonomic.

If C_1 is a subgroup of B containing $\zeta(B)$ then $F_1(C_1/\zeta(B))$ is commutative if and only if FC_1 is also commutative. Also W is torsion-free as FC_1 -module if and only if W_1 is torsion-free as $F_1(C_1/\zeta(B))$ -module.

Thus if C_2 is a subgroup of $B/\zeta(B)$ with F_1C_2 commutative then W_1 is F_1C_2 -torsion-free and the second condition for ‘holonomic’ is satisfied by W_1 .

It also follows that

$$\dim W_1 = m - \text{rk}(\zeta(B)). \quad (4.1)$$

By (1) of Proposition 4.1, there is a subgroup C_2 of B with $F_1(C_2/\zeta(B))$ commutative and

$$\dim W_1 = \text{rk}(B/\zeta(B)) - \text{rk}(C_2/\zeta(B)).$$

Thus FC_2 is commutative and so $\text{rk}(C_2) \leq m$. Thus

$$\dim W_1 = \text{rk}(B) - \text{rk}(C_2) \geq \text{rk}(B) - m. \quad (4.2)$$

Thus, combining (4.1) and (4.2), we have

$$\text{rk}(B) - m \leq m - \text{rk}(\zeta(B)). \quad (4.3)$$

It follows that $\text{rk}(B) + \text{rk}(\zeta(B)) \leq 2m$. But we have assumed the reverse inequality and so the inequalities in (4.2) and (4.3) are, in fact, equalities. In particular,

$$\dim(W_1) = \text{rk}(B) - m = (1/2)(\text{rk}(B) - \text{rk}(\zeta(B))) = (1/2) \text{rk}(B/\zeta(B)).$$

Thus W_1 is a strongly holonomic module. \square

Lemma 4.4. *Let C be a subgroup of A with FC commutative. Then C meets some carrier space subgroup of M trivially.*

Proof. By assumption M is torsion-free over FC . Thus any non-zero submodule, M_1 say, of M will also be torsion-free over FC and so will again have dimension m . In particular, we can take M_1 to be critical.

By Theorem 5.5 of [BG2], $(\pi_C(\Delta^*(M_1)))^* = \Delta^*(N)$ for some cyclic critical FC -submodule N of minimal dimension in M . Because M is FC -torsion-free, N must have dimension $\text{rk } C$; that is, $(\pi_C(\Delta^*(M_1)))^* = C^*$. Thus $\pi_C(\Delta^*(M_1)) = C^*$. But then, because a Euclidean space cannot be the union of finitely many proper subspaces, at least one of the m -dimensional convex polyhedra contained in $\Delta^*(M_1)$ must map onto C^* . That is, $\ker \pi_C + V = A^*$ for some carrier space V . Because V is a rational subspace of A^* , we have $V = \ker \pi_B$ for some (carrier space) subgroup B of A and so $C \cap B = \{1\}$. \square

Lemma 4.5. *If $\text{rk } A \geq 4$, then each carrier space of M has non-trivial intersection with some other carrier space.*

Proof. Let V be a carrier space of M and suppose that C is the corresponding carrier space subgroup. By Lemma 4.1, FC_1 is commutative for some subgroup C_1 of finite index in C . Thus M is torsion-free over FC_1 ; also the co-dimension of V is m which is at least 2. Thus, by Corollary 3.7 of

[BG3], V contains non-zero points which are non-generic for V . It follows from Corollary 3.5 that there are points $\chi \in V$ such that $LC_\chi(\Delta^*(M)) \not\subseteq V$. But then χ must lie in some other carrier space of V and so V intersects some other carrier space, as required. \square

4.2 Alternating bilinear maps on vector spaces

Our aim is to investigate the nature of the group generated by \bar{A} within DA . In particular, we have considerable amounts of information about the possible subgroups B of A for which DB is commutative. The structure is described by the commutator map from A to F . But rather than work with A and its subgroups, we shall work with the divisible hull $A \otimes \mathbb{Q}$ and its subspaces. The next paragraph translates the previous definitions to this context.

Let V and W be finite dimensional vector spaces over a field K and let $\phi : V \times V \rightarrow W$ be an alternating bilinear map. We shall often abbreviate $\phi(x, y)$ by (x, y) . We shall use terminology with group-theoretical overtones rather than that derived from the theory of forms on vector spaces. We shall say that, if $x, y \in V$ then x *centralises* y if $(x, y) = 0$ and if $S \subseteq V$ then the centraliser $\mathcal{C}(S)$ is the subspace of all those elements of V which centralise each element of S . The centraliser of V is called the *centre* of V . We shall say that a subspace U of V is *abelian* if any two elements of U centralise each other.

We now turn to the sort of structure we wish to establish for crossed products with a strongly holonomic module.

Definition 3. A symplectic base for V is a decomposition of V ,

$$V = \bigoplus_{i=0}^t V_i,$$

as a direct sum so that

1. V_0 is the centre of V ;
2. if $i \neq j$ then $(V_i, V_j) = \{0\}$;

3. (V_i, V_i) has dimension 1 and V_i has centre $\{0\}$;

4. if $i \neq j$ then $(V_i, V_i) \neq (V_j, V_j)$.

Observe that ϕ restricted to V_i for $i > 0$ yields a non-degenerate symplectic form on V_i and so the well-known properties of such a form hold. In particular, every non-zero element of V_i has centraliser, in V_i , of co-dimension one. Also, if A is an abelian subspace of maximal dimension in V_i with basis $\{x_1, \dots, x_m\}$ then V_i has a basis $\{x_1, \dots, x_m, y_1, \dots, y_m\}$ where $(y_i, y_j) = 0$ for each i, j and $(x_i, y_j) = 0$ precisely if $i \neq j$.

The definition of a symplectic base is designed to ensure a degree of uniqueness.

Lemma 4.6. *If V has a symplectic base (as above) then the subspaces $V_0 + V_i$ are uniquely determined up to re-arrangement. In particular, if the centre is trivial then the subspaces V_i are unique up to re-arrangement.*

Proof. This follows immediately from the fact that V_0 is the centre of V and that ‘the non-zero elements of $\cup_{i>0}(V_0 + V_i)$ are precisely the elements of V with centraliser of co-dimension at most one’. We prove the latter statement. Observe firstly that the elements of $V_0 + V_i$ certainly have co-dimension at most 1 since elements of V_0 have centraliser V and elements of V_i with $i \neq 0$ have centraliser in V_i of co-dimension 1.

If $v \in V$ then the function $\phi_v : V \rightarrow W$ given by $w \rightarrow \phi(v, w)$ is a linear map with kernel the centraliser of v . Thus the co-dimension of the centraliser of v is equal to the dimension of the image of ϕ_v . Let $v = \sum_{i=0}^t v_i$ with $v_i \in V_i$ and suppose, for example, that $v_1, v_2 \neq 0$. Then there exist $v'_1 \in V_1$ and $v'_2 \in V_2$ so that (v_i, v'_i) is non-zero for $i = 1, 2$. Thus (v, v'_1) and (v, v'_2) are non-zero elements of (V_1, V_1) and (V_2, V_2) respectively and so are independent. Thus at most one of the $v_i (i \geq 1)$ can be non-zero if the centraliser of v has co-dimension at most one. The proof is complete. \square

Proposition 4.7. *Suppose that V has a symplectic base $V = \bigoplus_i V_i$ with trivial centre and dimension $2m$. Let U be an abelian subspace of dimension at least m . Then*

$$U = \bigoplus_{i=1}^t (U \cap V_i).$$

Thus U has dimension exactly m and $U \cap V_i$ has dimension one half of the dimension of V_i .

Observe that it is easy to deduce a similar statement without the assumption of trivial centre by applying the proposition to V/V_0 .

Proof. We prove by induction on the dimension of V .

Case 1: suppose that U contains a non-zero element of some V_i .

Let us suppose that $0 \neq v \in V_1 \cap U$. Then there exists some $v' \in V_1$ so that $(v, v') \neq 0$. Set $X = \langle v, v' \rangle$. Then we can find a complement X_1 of X in V_1 so that $(X, X_1) = \{0\}$. It is easily checked that X_1, V_2, \dots, V_t forms a symplectic base for the sum $V' = X_1 + V_2 + \dots + V_t$. Further, $V = X \oplus V'$; let π be the projection of V onto X .

We claim that $\pi(U)$ has dimension 1. Otherwise, $\pi(U) = X$ and so $v' = \pi(u)$ for some $u \in U$. But then $(v, u) = 0$ as both v and u are in the abelian subspace U . However, $(v, u) = (v, \pi(u))$ as $v \in X$ and $u - \pi(u) \in V'$. Also $(v, \pi(u)) = (v, v') \neq 0$, a contradiction.

Thus $\pi(U)$ has dimension 1 and so $U_1 = U \cap \ker(\pi)$ has dimension at least $m - 1$. Since $V' = \ker(\pi)$ has dimension $2m - 2$, we can apply the inductive hypothesis to show that

$$U_1 = (U_1 \cap X_1) \oplus (U_1 \cap V_2) \oplus \dots \oplus (U_1 \cap V_t).$$

Also, as $v \notin \ker(\pi)$, $U = \langle v, U_1 \rangle$. Thus

$$\begin{aligned} U &= \langle v \rangle \oplus U_1 \\ &= \langle v \rangle \oplus (U_1 \cap X_1) \oplus (U_1 \cap V_2) \oplus \dots \oplus (U_1 \cap V_t) \\ &= (U \cap V_1) \oplus \dots \oplus (U \cap V_t). \end{aligned}$$

Thus the proof is complete in this case.

Case 2: suppose that $\pi_i(U)$ is a proper subspace of V_i where π_i is the projection onto V_i .

Let us suppose that $\pi_1(U) \neq V_1$. Then $\pi_1(U)$ has a non-zero centraliser in V_1 ; say $0 \neq v \in V_1$ centralises every element of $\pi_1(U)$. But then v centralises

every element of U and $\langle v, U \rangle$ is still abelian. But then Case 1 applied to $\langle v, U \rangle$ completes the proof.

Case 3: suppose that U contains an element u of the form $v_i + v_j$ where v_i and v_j are non-zero elements of V_i and V_j with $i \neq j$.

Using Case 2, we can suppose that $\pi_i(U) = V_i$ and so there is an element $u' \in U$ with $(v_i, \pi(u')) \neq 0$. But $(u, u') = 0$ as both u and u' are non-zero elements of the abelian subspace U and so

$$0 = (u, u') = (v_i + v_j, u') = (v_i, \pi_i(u')) + (v_j, \pi_j(u')).$$

Thus we also have that $(v_j, \pi_j(u'))$ is non-zero and that the (one-dimensional) subspaces (V_i, V_i) and (V_j, V_j) are equal. But this was prohibited in the definition of a symplectic base.

The general case. By Case 2, we can assume that $\pi_1(U) = V_1$. Let $\dim(V_1) = 2m_1$ and let $U_1 = U \cap \ker(\pi_1)$. Thus $\dim(U_1) = \dim(U) - 2m_1 \geq m - 2m_1$.

Choose an abelian subspace X_1 of dimension m_1 in V_1 and set $X = U \cap \pi_1^{-1}(X_1)$. Then $\dim X = \dim U - m_1 \geq m - m_1$.

Observe that, if $u \in U$ and $u = \pi_1(u)$ then $u \in V_1$ and so we can use Case 1 if $u \neq 0$. Thus the map $id - \pi_1$ is injective on U and so $Y = (id - \pi_1)(X)$ is a subspace of $\ker(\pi_1)$ having dimension at least $m - m_1$.

We claim that Y is abelian. For $i = 1, 2$ let $x_i \in X$ so that $y_i = x_i - \pi_1(x_i) \in Y$. Then

$$\begin{aligned} (y_1, y_2) &= (y_1, x_2) - (y_1, \pi_1(x_2)) = (y_1, x_2) \text{ as } y_1 \in \ker(\pi_1) \text{ and } \pi_1(x_2) \in V_1 \\ &= (x_1, x_2) - (\pi_1(x_1), x_2) = -(\pi_1(x_1), x_2) \text{ as } x_1, x_2 \in U \\ &= -(\pi_1(x_1), x_2) + (\pi_1(x_1), \pi_1(x_2)) \text{ as } \pi_1(x_i) \in X_1 \\ &= -(\pi_1(x_1), x_2 - \pi_1(x_2)) = 0 \text{ as above.} \end{aligned}$$

Thus Y is an abelian subspace of $\ker(\pi_1)$ and we can apply the inductive hypothesis to show that Y is the direct sum of the $Y \cap V_i$. In particular, $Y \cap V_2$ is non-zero; say $v \in V_2$ and $v \neq 0$ with $v \in Y$. So $v = x - \pi_1(x)$ with $x \in X$. But then $x \in U$ and $x = \pi_1(x) + v$ with $\pi_1(x) \in V_1$ and $v \in V_2$. This falls under one of Cases 1 or 3 and so completes the proof. \square

We now turn to establishing the existence of a symplectic base from the existence of 'sufficient' large abelian subspaces. The latter will arise in the

application because of the existence and size of the carrier space subgroups corresponding to strongly holonomic modules.

Suppose that $\dim V + \dim \zeta(V)$ is even; equal, say, to $2m$.

Definition 4. *We shall say that V has ample abelian subspaces if whenever X is a subspace of V containing an abelian subspace of dimension m with $\dim X + \dim \zeta(X) \geq 2m$ then equality holds and there exists a non-empty set Ω_X of m -dimensional abelian subspaces of X so that:*

1. *if $\dim(X/\zeta(X)) > 2$ then, given $U_1 \in \Omega_X$, there exists $U_2 \in \Omega_X$ with $U_1 \cap U_2 > \zeta(X)$;*
2. *given any abelian subspace U of X , there exists $U_1 \in \Omega_X$ such that $U \cap U_1 \leq \zeta(X)$.*

Observe that the subspaces X , which contain abelian subspaces of dimension m and satisfy the inequality, inherit the property of having ample abelian subspaces.

Proposition 4.8. *If V has ample abelian subspaces then V has a symplectic base.*

Proof. Observe that, if V has ample abelian subspaces, then so also does $V/\zeta(V)$ and if $V/\zeta(V)$ has a symplectic base then so also does V . Thus we can assume that the centre of V is zero. We shall use induction on the dimension of V .

Let U be any abelian subspace of V with dimension m . Then there exists $U_1 \in \Omega_V$ so that $U \cap U_1 = \{0\}$. There also exists $U_2 \in \Omega_V$ so that $U_1 \cap U_2$ is not zero; set $k = \dim(U_1 \cap U_2)$. Then, as the U_i are abelian of dimension m , we have that $X = U_1 + U_2$ has dimension $2m - k$ and centre at least $U_1 \cap U_2$. Thus $\dim X + \dim \zeta(X) \geq 2m$. Thus equality holds and X has ample abelian subspaces. Using the inductive hypothesis, X therefore has a symplectic base. Also, as

$$\dim X + \dim \zeta(X) = 2m = \dim X + \dim(U_1 \cap U_2),$$

it follows that $U_1 \cap U_2 = \zeta(X)$

Consider $U \cap X$. As $U_1 \leq X$ and $U \cap U_1 = \{0\}$, with both U and U_1 of dimension half that of V , it follows that $U + U_1 = V$ and so $U + X = V$. Thus $U \cap X$ has dimension $m - k$. Further, as $\zeta(X) = U_1 \cap U_2$, we have $(U \cap X) \cap \zeta(X) = \{0\}$. Choose a complement X' to the centre of X which contains $U \cap X$. Then X' has a symplectic base $X' = \bigoplus X_i$ with trivial centre and, by Proposition 4.7,

$$U \cap X = \bigoplus_i (U \cap X) \cap X_i.$$

Form a new abelian subspace U'_3 of X' by taking abelian subspaces in X_i which, for $i > 1$ complement $(U \cap X) \cap X_i$ and for $i = 1$, have dimension equal to that of $(U \cap X) \cap X_1$ but intersect it in dimension 1. The existence of such subspaces within the X_i follows easily from the fact that the restriction of ϕ to X_i is a non-degenerate form. Let $U_3 = U'_3 + \zeta(X)$. Then U_3 has dimension m and $U_3 \cap U = U_3 \cap (U \cap X)$ has dimension 1. Let $X' = U + U_3$. As before, we can show that X' has ample abelian subspaces and so the inductive hypothesis tells us that X' has a symplectic base with centre of dimension 1. Thus we have shown that every abelian subspace of dimension m lies in a subspace of dimension $2m - 1$ with centre of dimension 1 and a symplectic base.

Let V_1 and V_2 be two such subspaces of co-dimension 1 with centres X_1 and X_2 of dimension 1. Suppose that $(X_1, X_2) \neq \{0\}$. Set $V_3 = V_1 \cap V_2$ and $X_3 = X_1 + X_2$. Then $(V_3, X_3) = \{0\}$ and $V_3 \cap X_3 = \{0\}$ so that $V = V_3 \oplus X_3$. But $V_2 = V_3 \oplus X_2$ and so V_3 has a symplectic base. Hence so also does V .

We are left with the possibility that, whenever V_1 and V_2 are subspaces of co-dimension 1 with a symplectic base then their centres commute. That is, the subspace Y spanned by all centres of such subspaces of co-dimension one is abelian. But then, there is a subspace $U \in \Omega_V$ so that $Y \cap U = \{0\}$. We have shown, however, that U can be placed inside a subspace V' of co-dimension 1 with a symplectic base and that U must then contain the centre of V' and so intersect Y . Thus this case is not possible and the proof is complete. \square

4.3 Proof of Theorem 4.2

Let S be a free generating set for A and let $\bar{S} = \{\bar{a} : a \in S\}$ be its image in FA . Let N denote the multiplicative subgroup of FA generated by \bar{S} . Then the derived subgroup N' of N will lie in F and A will be isomorphic to N/N' .

Thus commutation in N will yield an alternating (\mathbb{Z} -)bilinear map $\hat{\phi} : A \times A \longrightarrow N'$. Set $V = A \otimes \mathbb{Q}$ and $W = N' \otimes \mathbb{Q}$. Then $\hat{\phi}$ extends to an alternating \mathbb{Q} -bilinear map $\phi : V \times V \longrightarrow W$. In the language of the previous section, we wish to show that V has a symplectic base. Thus we will need to show that V has ample abelian subspaces and we can then use Proposition 4.8.

Let X be any subspace of V satisfying $\dim X + \dim \zeta(X) \geq 2m$ and containing an abelian subspace of dimension m . Let B be the isolated subgroup A corresponding to subspace X . Then $\text{rk } B + \text{rk } \zeta(B) \geq 2m$. Thus we can apply Lemma 4.3 to show that equality holds and that $F_1(B/\zeta(B))$ has a holonomic module $M(B)$ (of dimension $m - \text{rk } \zeta(B)$). Let Ω'_X denote the complete inverse image under the map $B \rightarrow B/\zeta(B)$ of the set of carrier space subgroups of $\Delta^*(M(B))$. These will have rank equal to $\dim M(B) + \text{rk } \zeta(B) = m$. Further we may suppose $m \geq 2$, and if we apply Lemma 4.5 to B , we see that for each element C of Ω_X , there is another element C' so that $C + C'$ is of rank $2m$. Let Ω_X denote the subspaces spanned by the elements of Ω'_X . Then each element of Ω_X is supplemented by another element of Ω_X and, considering dimensions, we see that these two subspaces have trivial intersection. Thus condition (1) is satisfied in the requirement for ample abelian subspaces. Condition (2) follows immediately from Lemma 4.4.

Thus V has ample abelian subspaces and we can apply Proposition 4.8 to show that V has a symplectic base. Note that, because we assumed the centre of A to be trivial, then $\zeta(V)$ will also be trivial and we will have

$$V = V_1 \oplus \cdots \oplus V_t.$$

Let A_i be the isolated subgroup in A corresponding to V_i . Then the A_i will generate their direct product and this will have finite index in A . The remaining properties of the A_i follow immediately.

It remains to prove that finitely generated FA_i -modules of M are strongly holonomic. From Proposition 2.5 of [9], M has a critical submodule M_0 . Let N be a cyclic critical FA_i -submodule of M having minimal dimension. We claim that N is a holonomic module for FA_i .

Firstly, note that FA_i has centre F . Also observe that N is torsion-free as FB -submodule for any commutative FB , since a similar statement is true for M . Denote by $2m_i$ the rank of A_i . It remains to show that N has dimension m_i .

We can apply Theorem 5.5 of [9] to show that $\pi_i(\Delta^*(M))^* = \Delta^*(N)$ where π_i^* is the map $A^* \longrightarrow A_i^*$ induced by the injection $A_i \longrightarrow A$. Thus each carrier space of $\Delta^*(N)$ will be the image under π_i^* of a carrier space of $\Delta^*(M)$. We prove that, if V is any carrier space of $\Delta^*(M)$, then $\pi_i(V)$ has dimension m_i and it follows immediately that $\pi_i^*(N)$ has dimension m_i and so that N has dimension m_i .

Let B_0 denote the carrier space subgroup of A which is dual to V . Thus B has rank m and, by (2) of Proposition 4.1, B has a subgroup of finite index B_1 so that FB is commutative. But this implies, by Proposition 4.7, that B_1 has a subgroup of finite index of the form $\oplus(B_1 \cap A_i)$. Since B_1 has rank $m = m_1 + \cdots + m_t$ and each $B_1 \cap A_j$ has rank at most m_j , it follows that each $B_1 \cap A_j$ has rank exactly m_j . In particular, $B_1 \cap A_i$ has rank m_i . Thus $A_i/(B \cap A_i)$, and so also $(B + A_i)/B$ has rank m_i . Passing back to the dual, we see that $V/V \cap A_i^\circ$ has dimension m_i , where A_i° is dual to A_i ; that is, it is the kernel of π_i^* . Hence $\pi_i^*(V)$ has dimension m_i as required and the proof is complete.

5 Finitely presented abelian-by-nilpotent-of-class-two groups

We now aim to convert the results of the previous section into results about finitely presented abelian-by-nilpotent-of-class-two groups. Suppose that

$$\{1\} \longrightarrow M \longrightarrow G \longrightarrow H \longrightarrow \{1\}$$

with M abelian, H nilpotent of class 2 and with G finitely presented. Thus M is a finitely generated $\mathbb{Z}G$ -module. We shall write the operation of M as

addition.

We shall use the assumption of finite presentation for G in the form of the somewhat weaker consequence that the second homology $H_2(M, \mathbb{Z}) = M \wedge_{\mathbb{Z}} M$ is finitely generated when considered as a $\mathbb{Z}G$ -module via the diagonal action. There is an immediate problem in that we would like to pass to submodules M_1 of M but that the natural map $M_1 \wedge M_1 \rightarrow M \wedge M$ is, in general, not an injection. It is, however, an injection, in case either M and M_1 are torsion-free or M and M_1 are both of the same prime exponent p . In the former case we use the fact that M and M' are both flat \mathbb{Z} -modules and in the latter case, we observe that the exterior square over \mathbb{Z} is equal to the exterior square over $\mathbb{Z}/p\mathbb{Z}$. Combining this with the fact that epimorphisms of modules yield epimorphisms of exterior squares, we obtain the following.

Lemma 5.1. *Suppose that M_1 is a submodule of M and that $M \wedge M$ is finitely generated as $\mathbb{Z}H$ -module.*

1. *if M_1 is torsion-free, then $M_1 \wedge M_1$ is finitely generated as $\mathbb{Z}H$ -module;*
2. *if M_1 has prime exponent p and the p -torsion subgroup of M has bounded exponent then M_1 has a non-zero submodule M_2 so that $M_2 \wedge M_2$ is finitely generated as $\mathbb{Z}H$ -module .*

Proof. If M_1 is torsion-free then M_1 is isomorphic to a submodule of the quotient \overline{M} of M by its torsion subgroup. Then $\overline{M} \wedge \overline{M}$ is finitely generated and contains a submodule isomorphic to $M_1 \wedge M_1$. Since $\mathbb{Z}H$ is Noetherian, it follows that $M \wedge M$ is finitely generated

If M_1 has prime exponent p and if the p -torsion subgroup T of M has exponent p^l then $p^l M \cap T = \{0\}$ and so $p^l M \cap M_1 = \{0\}$. Choose k minimal so that $p^k M \cap M_1 = \{0\}$. Then M_1 is isomorphic to a submodule of $M/p^k M$. Let M_2 be the complete inverse image in M_1 of $\overline{M} = p^{k-1} M / p^k M$. The map $m \mapsto p^{k-1} m + p^k M$ is a homomorphism (of $\mathbb{Z}H$ -modules) and so $\overline{M} \wedge \overline{M}$ is finitely generated. As \overline{M} has exponent p and has a submodule isomorphic to M_2 , it follows that $M_2 \wedge M_2$ is finitely generated. \square

Lemma 5.2. *Let G be a finitely presented group with an abelian normal subgroup M with quotient which is nilpotent of class 2. Let M_1 be a non-trivial G -normal subgroup of M so that $M_1 \wedge M_1$ is finitely generated as*

$\mathbb{Z}G$ -module. Let C denote the centraliser of M_1 in G ; denote G/C by H and let Z denote the centre of H . Suppose:

1. $\text{Ann}_{\mathbb{Z}\mathbb{Z}}(M_1) = P$ is prime;
2. M_1 is torsion-free as $\mathbb{Z}\mathbb{Z}/P$ -module;
3. for some subgroup K of H with $Z \subseteq K$, M_1 is not torsion as $\mathbb{Z}K/(P.\mathbb{Z}K)$ -module.

Then

$$\text{rk}(K/Z) \leq \frac{1}{2} \text{rk}(H/Z).$$

Proof. As M_1 is not torsion as $\mathbb{Z}K/(P.\mathbb{Z}K)$ -module, there is $m \in M_1$ so that, if we set $V = m.\mathbb{Z}K$, then V is isomorphic to $\mathbb{Z}K/(P.\mathbb{Z}K)$. Set $M_2 = V.\mathbb{Z}H$. We wish to apply Lemma 9 of Segal [17] to V and M_2 . Set $J = \mathbb{Z}\mathbb{Z}/P$. By assumption, M_1 and so M_2 is torsion-free as J -module. Then Lemma 9 of [17] tells us that there is a non-zero ideal Λ of J so that if Q is any ideal of J which does not contain Λ then $M_2Q \cap V = VQ$.

As J is a finitely generated commutative domain, it follows from the Nullstellensatz that the Jacobson radical is trivial (see, for example, Section 4.5 of [13]) and so there is a maximal ideal Q_1 of J which does not contain Λ ; further J/Q_1 will be finite. Let Q be the (maximal) ideal of $\mathbb{Z}\mathbb{Z}$ so that $Q/P = Q_1$. Then $M_2Q \cap V = VQ$ and so M_2/M_2Q contains a copy of $V/VQ \cong \mathbb{Z}K/Q.\mathbb{Z}K$.

By Theorem G of Segal[17], M/MQ has Krull dimension at least that of the $\mathbb{Z}K$ -module V/VQ . But V/VQ is easily seen to be a crossed product of the (central) field $\mathbb{Z}\mathbb{Z}/Q$ with the group Z/K and a minor adaptation of the proof of Smith[18] for group rings shows that the Krull dimension of V/VQ equals $\text{rk}(K/Z)$. Thus

$$\text{kdim}(M_2/M_2Q) \geq \text{rk}(K/Z) \tag{5.1}$$

where $\text{kdim}(-)$ denotes Krull dimension and $\text{rk}(-)$ denotes the torsion-free rank or Hirsch length.

If B denotes the centraliser of M_2/M_2Q in H , then $M_1 \wedge M_1$ will be finitely generated as $\mathbb{Z}(H/B)$ -module. As $\mathbb{Z}\mathbb{Z}/Q$ is a finite integral domain,

it will have prime exponent as abelian group and so, we can apply Lemma 3 of Brookes[6] to obtain that

$$2 \operatorname{kdim}(M_2/M_2Q) \leq \operatorname{rk}(H/B). \quad (5.2)$$

As $\mathbb{Z}Z/Q$ is finite, $B \cap Z$ must have finite index in Z and so $\operatorname{rk}(H/B) = \operatorname{rk}(H/BZ) \leq \operatorname{rk}(H/Z)$. Hence, combining this with (5.1) and (5.2), we obtain that

$$2h(K/Z) \leq \operatorname{rk}(H/Z)$$

as required. \square

We now aim to translate the results of the previous section into a result for groups. We begin with a special case.

Proposition 5.3. *Suppose the sequence of groups*

$$\{1\} \longrightarrow M \longrightarrow G \longrightarrow H \longrightarrow \{1\}$$

is exact with M abelian, H torsion-free nilpotent of class 2 and G finitely presented. Suppose that M_1 is a non-zero G -normal subgroup of M with $M_1 \wedge M_1$ finitely generated as G -module. Then there exists a subgroup G_0 of finite index in G and a non-zero G_0 -normal subgroup N of M_1 so that the quotient of G by the centraliser of N is a central product of groups which are either cyclic or Heisenberg.

Proof. Let Z denote the centre of H and consider M_1 as $\mathbb{Z}Z$ -module. Let P be a maximal associated prime of M_1 and let N_1 be the victim of P in N ; that is, $N = \{n \in N_1 : nP = 0\}$. Then N is a $\mathbb{Z}H$ -submodule of N_1 and is torsion-free as $\mathbb{Z}Z/P$ -module. As abelian group, N must either be torsion-free or of prime exponent p and so we can apply Lemma 5.1 to show that N_1 has a non-zero submodule N with $N \wedge N$ finitely generated as H -module. It now follows from Proposition 2 of [5] that N is an impervious $\mathbb{Z}H$ -module.

Let F be the field of fractions of $\mathbb{Z}Z/P$. As N is torsion-free as $\mathbb{Z}Z/P$ -module, it will embed into $\widehat{N} = N \otimes_{\mathbb{Z}Z/P} F$ and the latter has a natural structure as $\mathbb{Z}H/(P.\mathbb{Z}H) \otimes_{\mathbb{Z}Z/P} F$ -module. But it is easy to check that this ring has a natural structure as a crossed product of the central field F by the

free abelian group of finite rank $A = H/Z$. Thus we now have an FA -module \widehat{N} .

If \widehat{N} has dimension d then there is a subgroup B of A of rank d so that \widehat{N} is not torsion as FB -module. If K is the subgroup of H so that $K/Z = B$, then it is easily verified that N is not torsion as $\mathbb{Z}K/P.\mathbb{Z}K$ -module. But then, by Lemma 5.2, $d = \text{rk}(K/Z) \leq (1/2) \text{rk}(H/Z) = \text{rk}(A)$; that is

$$\dim(\widehat{M}) \leq \frac{1}{2} \text{rk}(A).$$

It is straightforward to check (see for example, the proof of Theorem 3.1 of [BG4]) that \widehat{N} is also impervious as FA -module. It is also clear that since F is constructed from the centre of H , it will be the centre of FA . Then, by Corollary 5 of Brookes[7], $\dim(\widehat{N}) \geq \frac{1}{2} \text{rk}(A)$ and so $\dim(\widehat{N}) = \frac{1}{2} \text{rk}(A)$. We also have, from Theorem 4 of Brookes[7], that \widehat{N} is FB -torsion-free for any F -abelian subgroup B of A . Thus \widehat{N} is a holonomic FA -module. It now follows from Theorem 4.2 that A has a subgroup of finite index of the form $\oplus A_i$ where each A_i generates a Heisenberg group within FA and these Heisenberg groups commute.

Let C denote the centraliser of N within H . Then there exists subgroups H_i of H so that H_i/C is Heisenberg, the subgroups H_i/C are pairwise commuting and so that the subgroups $H_i Z/C$ generate a subgroup of finite index in H . Thus $\prod H_i Z/C$ is of finite index in H/C and so H/C has a subgroup of finite index which is a central product of subgroups which are either Heisenberg or cyclic. \square

Recall that a submodule is *essential* if it is non-zero and has non-zero intersection with every non-zero submodule.

Lemma 5.4. *Let H be a finitely generated nilpotent group and let M be a finitely generated $\mathbb{Z}H$ -module. If N is an essential submodule of M and K is a normal subgroup of H which acts nilpotently on N , then K acts nilpotently on M .*

Proof. Let J denote the kernel of the natural map $\mathbb{Z}H \rightarrow \mathbb{Z}(H/K)$. Because K acts nilpotently on N , we have $NJ^m = \{0\}$ for some m . Set $I = J^m$. Because I is an ideal of the group ring of a nilpotent group, it satisfies the

weak Artin-Rees condition; that is, there is an integer n so that $MI^n \cap N = NI$ (see Theorem 11.3.11 and Theorem 11.2.8 of Passman[15]). But $NI = \{0\}$ and N is essential. Thus $MI^n = MJ^{mn} = \{0\}$ and so K acts nilpotently on M . \square

Theorem 5.5. *Let G be a finitely presented group with a normal abelian subgroup M so that $H = G/M$ is torsion-free nilpotent of class 2. Then there exist normal subgroups M_i of G , lying in M , and normal subgroups C_i of H so that*

1. H/C_i is virtually a central product of Heisenberg and cyclic groups;
2. C_i acts nilpotently on M_i ;
3. if $C = \cap_i C_i$ then C acts nilpotently on M .

Proof. We shall consider M as $\mathbb{Z}H$ -module. Since $\mathbb{Z}H$ is Noetherian, we can find a maximal finite direct sum of non-zero submodules M_i of M so that each M_i is uniform. Then the sum M' of the M_i is necessarily essential.

Each M_i necessarily has a non-zero submodule which is either of prime exponent or is torsion-free and, by Lemma 5.1, therefore has a non-zero submodule N'_i so that $N'_i \wedge N'_i$ is finitely generated as $\mathbb{Z}G$ -submodule. Thus, by Proposition 5.3, there is a subgroup H_i of finite index in H and a non-zero $\mathbb{Z}H$ -submodule N_i of N'_i so that if C_i denotes the centraliser of N_i then H/C_i is a central product of groups which are either cyclic or Heisenberg.

But N_i is a non-zero submodule of the uniform module M_i and so is essential. Thus, by Lemma 5.4, C_i acts nilpotently on M_i . Thus, if C is the intersection of the C_i , then C acts nilpotently on the sum M' of the M_i . But M' is an essential submodule of M and so C acts nilpotently on M . \square

Corollary 5.6. *Let G be a finitely presented groups which is an extension of an abelian normal subgroup by a group which is torsion-free nilpotent of class 2. Suppose that any two non-trivial normal subgroups of G have non-trivial intersection. Let F denote the Fitting subgroup of G . Then G/F has a subgroup of finite index which is a central product of groups which are cyclic or Heisenberg.*

Proof. This is simply the case of Theorem 5.5 in which one of the subgroups M_i can be taken equal to M . \square

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